

Friedel oscillations in quantum degenerate collisional plasma. Screening of point charge

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Research of influence of collisions on Friedel oscillations in quantum degenerate collisional plasma ($T = 0$) is carried out for the first time. It is shown that presence of collisions in plasma leads to exponential decreasing of amplitude and phase shift of Friedel oscillations. In linear approximation the phase shift is equal to the half of quantity inverse to product of Fermi's wave number by free length path of electrons. The correct expression for longitudinal dielectric permeability of the quantum collisional plasma found by the authors (see arXiv:1001.3937 [math-ph] 22 Jan 2010) is used.

Key words: degenerate collisional plasma, dielectric permeability, Friedel oscillations, Kohn singularities, screening of point charges.

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1. Introduction

Under classical consideration of degenerate electronic plasma potential $V(r)$ around point charge Q is described by the classical formula of Thomas — Fermi screening [1]

$$V(r) = \frac{Q}{r} \exp(-k_{TF}r).$$

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Here k_{TF} is the inverse screening radius of Thomas — Fermi

$$k_{TF} = \left(\frac{6\pi n e^2}{\varepsilon_F} \right)^{1/2},$$

where e is the electron charge, n is the electron concentration, ε_F is the Fermi energy.

Fridel was the first [2] – [6] who has found out that asymptotic (on the large distances) decreasing of screening potential of point charge under quantum consideration of degenerate collisionless plasma has not only monotonously decreasing, but also oscillatory character. The reason of such oscillations is sharp falling (to zero) of Fermi distributions for electrons $f_F(v)$ behind Fermi's surface ,

$$f_F(v) = \Theta(v_F - v),$$

where $\Theta(x)$ is the Heaviside function,

$$\Theta(x) = \begin{cases} 1, & x > 0, \\ 0, & x < 0. \end{cases}$$

This singularity of Fermi distribution leads to so called Kohn singularities (see [7] – [13]). Kohn singularities are consequence of logarithmic singularities of the longitudinal dielectric permeability of degenerate plasma. Just Kohn singularities lead us to Friedel oscillations in degenerate plasma.

In the work [13] the authors found out the dependence of Friedel's oscillations on temperature in collisionless non-degenerate plasma. It is shown that under non-zero temperature Friedel oscillations amplitude decreases exponentially with distance.

In the present work the research of influence of collisions on Friedel oscillations in quantum degenerate collisional plasma ($T = 0$) is carried out. The correct expression for longitudinal dielectric permeability of the quantum collisional plasma found by authors (see [14]) is used. It

is shown that the presence of collisions in plasma results in exponential decreasing of amplitude and phase shift of Friedel oscillations.

In linear approximation phase shift is equal to half of the quantity inverse to product of the Fermi wave number by free length path of electrons. In more details, phase shift is proportional to effective frequency of electron collisions with plasma particles. From here it is clear that when the effective frequency of collisions decreases to zero (plasma becomes collisionless), phase shift tends to zero. It is agreed with classical result [11], [12].

Let's emphasize that we use the expression for dielectric longitudinal permeability of collisional degenerate plasma found by us in [14]. This expression is deduced on the basis of the solution of the quantum kinetic equation for Wigner function in coordinate space, instead of momentum space as Mermin [15]. Mermin in [15] has received the expression for longitudinal dielectric permeability in the quantum collisional plasma using the kinetic equation in momentum space. However, if we use the Mermin expression for longitudinal permeability, then exponential decreasing of amplitude of Friedel oscillations cannot be found out. As the reason of this fact is that the Mermin longitudinal permeability for a static case does not depend on collision frequency.

2. Potential of point charge

Let us consider a point charge Q in the origin of coordinates. Then the equation describing behaviour of potential V round the point charge has the form

$$\Delta V(\mathbf{r}) = -4\pi\rho(\mathbf{r}) - 4\pi Q\delta(\mathbf{r}). \quad (1.1)$$

Here Δ is the Laplace operator, $\delta(\mathbf{r})$ is the Dirac delta function of vector argument, $\delta(\mathbf{r}) = \delta(x)\delta(y)\delta(z)$; $\rho(\mathbf{r})$ is the density of induced charge.

We use representation of quantities in (1.1) in the form of Fourier integrals

$$\begin{aligned}\delta(\mathbf{r}) &= \frac{1}{(2\pi)^3} \int e^{i\mathbf{k}\mathbf{r}} d^3k, \\ V(\mathbf{r}) &= \frac{1}{(2\pi)^3} \int e^{i\mathbf{k}\mathbf{r}} V_{\mathbf{k}} d^3k, \\ \rho(\mathbf{r}) &= \frac{1}{(2\pi)^3} \int e^{i\mathbf{k}\mathbf{r}} \rho_{\mathbf{k}} d^3k.\end{aligned}$$

Then from the equation (1.1) we receive

$$k^2 V_{\mathbf{k}} = 4\pi \rho_{\mathbf{k}} + 4\pi Q. \quad (1.2)$$

Earlier we have shown [14] that

$$\rho_{\mathbf{k}} = \frac{\mathbf{k}\mathbf{j}_{\mathbf{k}}}{\omega}, \quad \rho_{\mathbf{k}} = \frac{\sigma_l(\mathbf{k})\mathbf{k}\mathbf{E}_{\mathbf{k}}}{\omega} = -\frac{i\sigma_l(\mathbf{k})k^2}{\omega} V_{\mathbf{k}}.$$

Here $\sigma_l(\mathbf{k})$ is the longitudinal electric conductivity of quantum plasma found in work [14].

Substituting this relation into (1.2) we receive

$$k^2 \left(1 + \frac{4\pi i \sigma_l(\mathbf{k})}{\omega} \right) V_{\mathbf{k}} = 4\pi Q,$$

or

$$k^2 \varepsilon_l(\mathbf{k}) V_{\mathbf{k}} = 4\pi Q,$$

where $\varepsilon_l(\mathbf{k})$ is the longitudinal dielectric permeability of quantum plasma,

$$\varepsilon_l(\mathbf{k}) = 1 + \frac{4\pi i \sigma_l(\mathbf{k})}{\omega}.$$

Thus spectral density of electric potential is expressed in terms of the quantity of charge

$$V_{\mathbf{k}} = \frac{4\pi Q}{k^2 \varepsilon_l(k)}. \quad (1.3)$$

When deducing (1.3) we used the fact that dielectric permeability depends only on the module \mathbf{k} , $\varepsilon_l(\mathbf{k}) = \varepsilon_l(k)$. It is obvious that the following relation relation $V(\mathbf{r}) = V(r)$ also takes place.

Then for the quantity $V(r)$ we have

$$V(r) = \frac{4\pi Q}{(2\pi)^3} \int \frac{e^{i\mathbf{k}\mathbf{r}}}{k^2 \varepsilon_l(k)} d^3k = \frac{4\pi Q}{(2\pi)^2} \int \frac{e^{ikr \cos \theta}}{\varepsilon_l(k)} \sin \theta d\theta dk.$$

Integrating by the angular variable θ we receive

$$V(r) = \frac{1}{r} \cdot \frac{2Q}{\pi} \int_0^\infty \frac{\sin(kr)}{\varepsilon_l(k)k} dk. \quad (1.4)$$

The formula (1.4) corresponds to the formula received in the monograph of Harrison (see [11], p. 334) if we present

$$V_q^0 = \frac{Q}{4\pi q^2}.$$

Let's consider asymptotic behaviour of potential $V(r)$, $r \rightarrow \infty$. It is possible to consider electric field to be weak enough in this area. Therefore linear approximation is applicable here. Fridel oscillations appear because of logarithmic singularities of dielectric permeability.

After double integration of expression (1.4) by parts we find [11]

$$V(r) = \frac{1}{r^3} \cdot \frac{2Q}{\pi} \int_0^\infty \frac{\sin(rk) \varepsilon_l''(k)}{k \varepsilon_l^2(k)} dk + O(e^{-k_{TF}r}),$$

or, believing $k = qk_F$,

$$V(r) = \frac{1}{r^3} \cdot \frac{2Q}{k_F^2 \pi} \int_0^\infty \frac{\sin(rk_F q) \varepsilon_l''(q)}{\varepsilon_l^2(q)} dq + O(e^{-k_{TF}r}). \quad (1.5)$$

In (1.5) we have left in explicit form only that term which results in oscillations of potential. Other item designated as $O(e^{-k_{TF}r})$ quickly decreases in exponential way, on the Thomas — Fermi radius it decays with distance.

Integrals of such kind as (1.5) usually tend to zero at tendency of distance r to infinity because of fast oscillations $\sin rk$. However as it will

be seen the second derivative of $\varepsilon_l''(k)$ contains singular Cauchy kernel that gives the non-zero contribution to integral. Friedel oscillations are caused by logarithmic singularity of dielectric permeability.

Function standing under the sign of integral in (1.5) is even, therefore we can expand the integration onto whole real axis:

$$V(r) = \frac{1}{r^3} \cdot \frac{Q}{\pi} \int_{-\infty}^{\infty} \frac{\sin(rk_F q) \varepsilon_l''(q)}{q \varepsilon_l^2(q)} dq + O(e^{-k_{TF} r}), \quad r \rightarrow \infty. \quad (1.6)$$

Let us recall that under positive values of $q > 0$ the quantity q equals to $q = \frac{|\mathbf{k}|}{k_F}$, and under negative values the quantity q has not any physical meaning.

3. Friedel oscillations

In the expression (1.6) there is an expression of the longitudinal dielectric permeability of quantum degenerate collisional plasma $\varepsilon_l(q)$. According to the work [14] this expression has the following form for statical case $\omega = 0$ considered here

$$\varepsilon_l(q) = 1 + \frac{3x_p^2}{2q^2} \frac{1 - g_+(q) + g_-(q)}{1 - iyg_0(q)}. \quad (2.1)$$

In (2.1) the following dimensionless parameters are introduced

$$x_p = \frac{\omega_p}{k_F v_F}, \quad y = \frac{\nu}{k_F v_F} = \frac{1}{lk_F}, \quad l = v_F \tau,$$

where ω_p is the plasma (Langmuir) frequency, $k_F = p_F/\hbar$ is the Fermi wave number, $p_F = mv_F$ is the Fermi momentum, v_F is the Fermi velocity, m is the electron mass, ν is the effective electron frequency, $l = v_F/\nu$ is the electron free length path, $\tau = 1/\nu$ is the time of electron free length path.

These functions $g_0(q)$, $g_+(q)$, $g_-(q)$ in statical limit ($\omega \rightarrow 0$) have the following form

$$g_0(q) = \frac{1}{2q} \ln \frac{iy + q}{iy - q},$$

$$g_+(q) = \frac{1}{8q^3} \left[(q^2 + 2iy)^2 - 4q^2 \right] \ln \frac{q^2 + 2q + 2iy}{q^2 - 2q + 2iy},$$

$$g_-(q) = \frac{1}{8q^3} \left[(q^2 - 2iy)^2 - 4q^2 \right] \ln \frac{q^2 - 2q - 2iy}{q^2 + 2q - 2iy}.$$

Let's find roots of the equations

$$q^2 \pm 2q \pm 2iy = 0.$$

We have

$$q_{1,2} = \mp 1 \pm \sqrt{1 \pm 2iy}. \quad (2.2)$$

Further let's consider the quantity y as small parameter and designate it through ε

$$\varepsilon \equiv y = \frac{\nu}{k_F v_F} = \frac{\nu \hbar}{m v_F^2} = \frac{\nu \hbar}{2\varepsilon_F}.$$

In linear approximation by ε for roots of (2.2) we receive

$$q_{1,2} = \mp 1 \pm (1 \pm i\varepsilon).$$

We present the functions $g_+(q)$ and $g_-(q)$ in the following form

$$g_+(q) = \frac{q^2 + \varepsilon^2}{8q^3} (q + 2 - i\varepsilon)(q - 2 + i\varepsilon) \ln \frac{(q + i\varepsilon)(q + 2 - i\varepsilon)}{(q - i\varepsilon)(q - 2 + i\varepsilon)},$$

$$g_-(q) = \frac{q^2 + \varepsilon^2}{8q^3} (q - 2 - i\varepsilon)(q + 2 + i\varepsilon) \ln \frac{(q + i\varepsilon)(q - 2 - i\varepsilon)}{(q - i\varepsilon)(q + 2 + i\varepsilon)},$$

or, in linear approximation, rejecting terms proportional to ε^2 , we have

$$g_+(q) = \frac{1}{8q} g_1(q), \quad g_-(q) = \frac{1}{8q} g_2(q),$$

where

$$g_1(q) = (q - 2 + i\varepsilon) f_{+-}(q) - (q + 2 - i\varepsilon) f_{-+}(q)$$

$$g_2(q) = (q + 2 + i\varepsilon) f_{--}(q) - (q - 2 - i\varepsilon) f_{++}(q).$$

Here the functions containing Kohn singularities are introduced

$$f_{++}(q) = (q + 2 + i\varepsilon) \ln(q + 2 + i\varepsilon),$$

$$\begin{aligned}
f_{--}(q) &= (q - 2 - i\varepsilon) \ln(q - 2 - i\varepsilon), \\
f_{+-}(q) &= (q + 2 - i\varepsilon) \ln(q + 2 - i\varepsilon), \\
f_{-+}(q) &= (q - 2 + i\varepsilon) \ln(q - 2 + i\varepsilon).
\end{aligned}$$

The second derivatives of these functions result in Cauchy kernels

$$f''_{\pm\pm}(q) = \frac{1}{q \pm 2 \pm i\varepsilon}.$$

Let's find the second derivative $\varepsilon''_l(q)$. In the expression for $\varepsilon''_l(q)$ we leave only those items, which contain Kohn singularities and result in the Friedel oscillations

$$\varepsilon''_l(q) = -\frac{3x_p^2}{2q^2} \frac{g''_+(q) - g''_-(q)}{1 - g_0(q)} = -\frac{3x_p^2}{16q^3} \frac{g''_1(q) - g''_2(q)}{1 - g_0(q)}. \quad (2.3)$$

Let's return to the integral (1.6) and by means of (2.3) we present it in the form

$$V(r) = -\frac{1}{r^3} \cdot \frac{3Qx_p^2}{16\pi} \int_{-\infty}^{\infty} \frac{[g''_1(q) - g''_2(q)] \sin(k_F r q)}{q^4 \varepsilon_l^2(q) [1 - g_0(q)]} + O(e^{-k_{TF} r}), \quad r \rightarrow \infty. \quad (2.4)$$

Let's show that in a static limit ($\omega \rightarrow 0$) the expression $\varepsilon_l(q)$ is real. Really, according to the results from [14] the dielectric permeability is expressed by equality

$$\varepsilon_l(q) = 1 + \frac{3x_p^2}{4y^2} \cdot \frac{\int_{-1}^1 \frac{(1 - t^2) dt}{(1 - i\omega\tau + iqt/y)^2 + q^4/(4y^2)}}{1 - \frac{1}{2} \int_{-1}^1 \frac{dt}{1 - i\omega\tau + iqt/y}}.$$

We receive from here in the static limit

$$\varepsilon_l(q) = 1 + \frac{3x_p^2}{4} \cdot \frac{\int_{-1}^1 \frac{(1 - t^2) dt}{(iqt + y)^2 + q^4/4}}{1 - \frac{y}{2} \int_{-1}^1 \frac{dt}{iqt + y}}.$$

After simple transformations we reduce the previous expression to the form which does not contain imaginary unit

$$\varepsilon_l(q) = 1 + \frac{3x_p^2}{4} \cdot \frac{\int_{-1}^1 \frac{(1-t^2)(q^4/3 + y^2 - q^2 t^2) dt}{(q^4/4 + y^2 - q^2 t^2)^2 + 4q^2 y^2 t^2}}{1 - \frac{y^2}{2} \int_{-1}^1 \frac{dt}{q^2 t^2 + y^2}}.$$

In the expression $g_1''(q) - g_2''(q)$ from the equality (2.4) we will leave the terms containing Kohn singularities and leading to Friedel oscillations again. As the result we receive

$$\begin{aligned} g_1''(q) - g_2''(q) &= \\ &= \frac{q-2+i\varepsilon}{q+2-i\varepsilon} - \frac{q+2-i\varepsilon}{q-2+i\varepsilon} - \frac{q+2+i\varepsilon}{q-2-i\varepsilon} + \frac{q-2-i\varepsilon}{q+2+i\varepsilon}. \end{aligned}$$

Now integral (2.4) we will present in the following form

$$\begin{aligned} V(r) = -\frac{1}{r^3} \cdot \frac{3Qx_p^2}{16\pi} \int_{-\infty}^{\infty} \frac{\sin(k_F r q)}{q^4 \varepsilon_l^2(q) [1 - g_0(q)]} &\left[\frac{q-2+i\varepsilon}{q+2-i\varepsilon} - \frac{q+2-i\varepsilon}{q-2+i\varepsilon} - \right. \\ &\left. - \frac{q+2+i\varepsilon}{q-2-i\varepsilon} + \frac{q-2-i\varepsilon}{q+2+i\varepsilon} \right] dq. \end{aligned} \quad (2.5)$$

Let's designate now

$$\varphi(q) = \frac{q+(2+i\varepsilon)}{q-(2+i\varepsilon)} + \frac{q+(2-i\varepsilon)}{q-(2-i\varepsilon)}.$$

It is obvious that other two terms in the square brackets are equal to $\varphi(-q)$:

$$\varphi(-q) = \frac{q-(2+i\varepsilon)}{q+(2+i\varepsilon)} + \frac{q-(2-i\varepsilon)}{q+(2-i\varepsilon)}.$$

Now the expression (2.5) we will write in the form

$$V(r) = \frac{1}{r^3} \cdot \frac{3Qx_p^2}{16\pi} \int_{-\infty}^{\infty} \frac{\varphi(q) - \varphi(-q)}{q^4 \varepsilon_l^2(q) [1 - g_0(q)]} \sin(k_F r q) dq.$$

This equality can be simplified

$$V(r) = \frac{1}{r^3} \cdot \frac{3Qx_p^2}{8\pi} \int_{-\infty}^{\infty} \frac{\varphi(q) \sin(k_F r q)}{q^4 \varepsilon_l^2(q) [1 - g_0(q)]} dq. \quad (2.6)$$

Let's notice that expression $q^4 \varepsilon_l^2(q)$ does not vanish in the point $q = 0$. Besides, we will notice that in the linear approximation of $1 - g_0(q) = 1$. Really, we multiply the subintegral function from expression for $g_0(q)$ by fraction, numerator and denominator of which is the expression, conjugating to the denominator from $g_0(q)$, we receive:

$$\begin{aligned} g_0(q) &= \frac{i\varepsilon}{2} \int_{-1}^1 \frac{(qt + iy) dt}{q^2 t^2 + \varepsilon^2} = \\ &= \frac{iyq}{2} \int_{-1}^1 \frac{t dt}{q^2 t^2 + \varepsilon^2} + \frac{\varepsilon^2}{2} \int_{-1}^1 \frac{dt}{q^2 t^2 + \varepsilon^2} = O(\varepsilon^2) \quad (\varepsilon \rightarrow 0), \end{aligned}$$

as it was required to show.

Now the integral from (2.6) we will write down in an explicit form

$$V(r) = \frac{1}{r^3} \cdot \frac{3Qx_p^2}{8} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \left[\frac{q + (2 + i\varepsilon)}{q - (2 + i\varepsilon)} + \frac{q + (2 - i\varepsilon)}{q - (2 - i\varepsilon)} \right] \frac{e^{ik_F r q} - e^{-ik_F r q}}{q^4 \varepsilon_l^2(q)} dq.$$

In last integral subintegral expression has two singular Cauchy kernels, equal to infinity in the complex conjugate points $q = 2 \pm i\varepsilon$. Integration near the point $q = 2$ gives the basic contribution to the expression for potential. For the approximate calculus of last integral we use the method stated, for example, in the monograph [11].

For this purpose we calculate continuous functions from subintegral expression in the point $q = 2$:

$$\left[q^4 \varepsilon_l(q) \right] \Big|_{q=2} = 16 \left(1 + \frac{3}{8} x_p^2 \right)^2.$$

We receive that

$$V(r) = \frac{1}{r^3} \frac{3Qx_p^2}{2(8+3x_p^2)^2} \times \\ \times \int_{-\infty}^{\infty} \left[\frac{q+(2+i\varepsilon)}{q-(2+i\varepsilon)} + \frac{q+(2-i\varepsilon)}{q-(2-i\varepsilon)} \right] \left[e^{ik_F r q} - e^{-ik_F r q} \right] dq.$$

The integral from the previous equality is equal to the sum of residues relatively the simple poles in points $q = 2 \pm i\varepsilon$:

$$V(r) = \frac{1}{r^3} \cdot \frac{3Qx_p^2}{(8+3x_p^2)^2} \left[(2+i\varepsilon)e^{ik_F r(2+i\varepsilon)} + (2-i\varepsilon)e^{-ik_F r(2-i\varepsilon)} \right].$$

From here we obtain

$$V(r) = \frac{1}{r^3} \cdot \frac{6Qx_p^2 e^{-k_F r \varepsilon}}{(8+3x_p^2)^2} \left[2 \cos 2k_F r - \varepsilon \sin 2k_F r \right].$$

Finally we get

$$V(r) = A \frac{e^{-k_F r \varepsilon}}{r^3} \cos(2k_F r + \varphi),$$

or

$$V(r) = A \frac{e^{-k_F r y}}{r^3} \cos\left(2k_F r + \frac{y}{2}\right), \quad (2.9)$$

Here

$$\varphi = \arctg \frac{\varepsilon}{2} = \frac{\varepsilon}{2} = \frac{y}{2},$$

$$A = \frac{12Qx_p^2}{(8+3x_p^2)^2}.$$

The expression (2.9) can be rewritten in dimensional parameters

$$V(r) = A \frac{e^{-\frac{\nu}{k_F} r}}{r^3} \cos\left(2k_F r + \frac{\nu}{2k_F v_F}\right).$$

From this formula at $\nu = 0$ the classical result for collisionless plasma is received (see, for example, [11])

$$V(r) = \frac{A}{r^3} \cos(2k_F r).$$

Let's designate through $R = k_F r$ dimensionless length, and introduce dimensionless frequency of collisions $y = \varepsilon = \frac{\nu}{k_F v_F}$. Then the formula (2.9) for potential will be written in the form

$$V(R) = A k_F^3 \frac{e^{-Ry}}{R^3} \cos(2R + \frac{y}{2}). \quad (2.10)$$

Graphical study of Friedel oscillations we will carry out for the case, when $Ak_F^3 = 10^5$ (fig. 1 - fig. 3 see). From fig. 3 it is visible that with growth of frequencies of electron collisions the amplitude of Friedel oscillations decreases.

4. Conclusion

In the present work we use the expression for dielectric permeability of quantum degenerate collisional plasma found by the authors earlier [14]. With the help of it the research of influence of collisions on Friedel oscillations is carried out for the first time.

It is shown that presence of collisions in plasma results in exponential decrease of amplitude of Friedel oscillations and to inverse shift of the phase oscillations. The logarithmic decrement of decrease in linear approximation is equal to $r = \frac{v_F}{\nu}$, and the inverse phase shift is equal to $\frac{\nu}{2k_F v_F} = \frac{1}{2k_F l}$, where l is the mean free path of electrons.

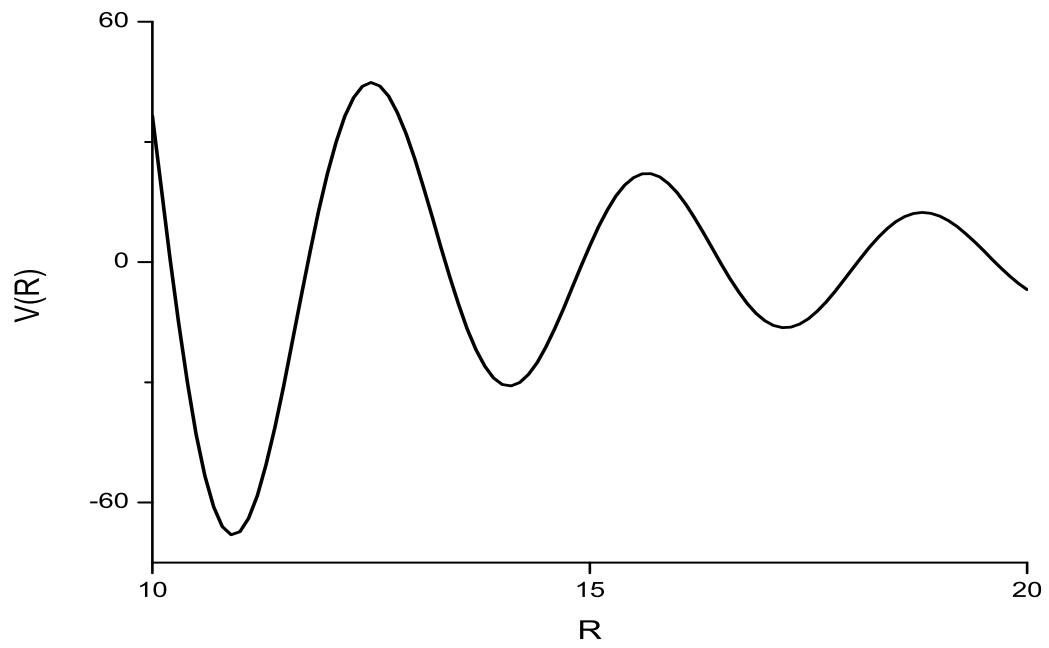


Figure 1: Friedel oscillations in the case $y = 10^{-2}$, $10 < R < 20$.

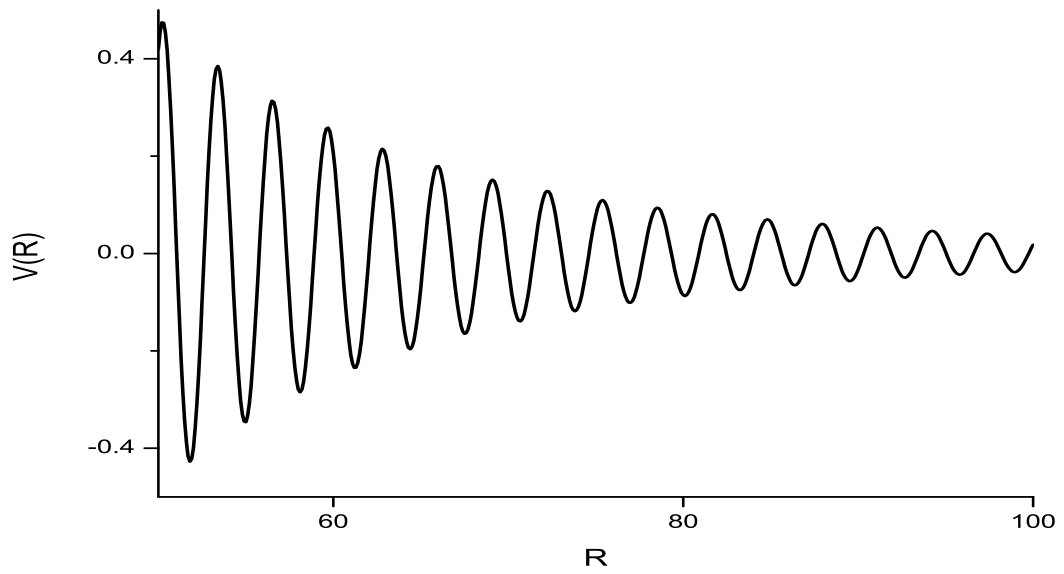


Figure 2: Friedel oscillations in the case $y = 10^{-2}$, $50 < R < 100$.

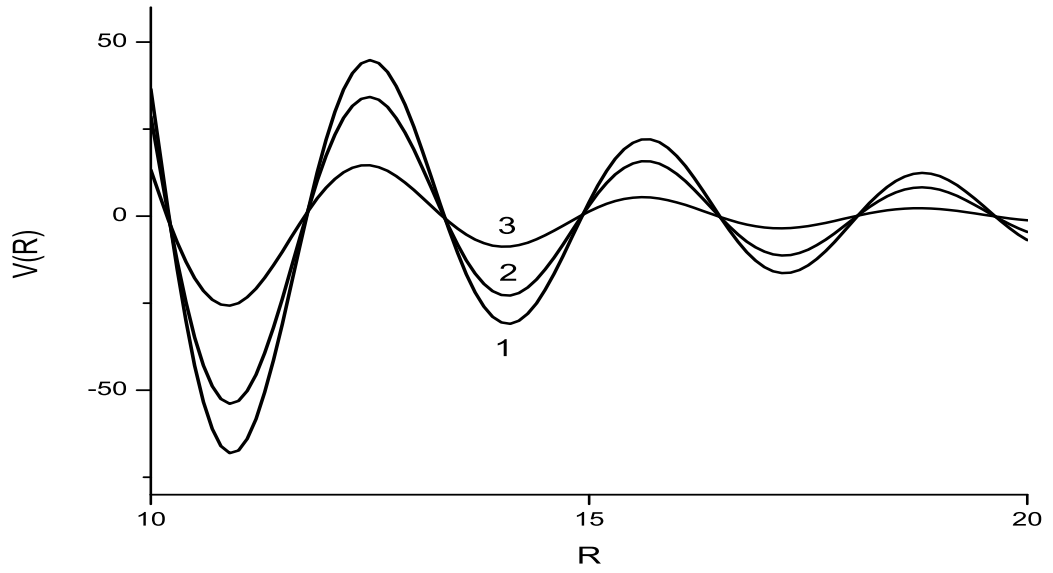


Figure 3: Friedel oscillations in the case $y = 10^{-2}$, $10 < R < 20$. Curves of 1, 2, 3 correspond to the values of parameter y , $y = 10^{-2}, 10^{-1.5}, 10^{-1}$.

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